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ON CHARACTERIZATIONS OF THE POWER DISTRIBUTION VIA THE IDENTICAL HAZARD RATE OF LOWER RECORD VALUES

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Abstract. In this article, we present characterizations of the power distribution via the identical hazard rate of lower record values that X_n has the power distribution if and only if for some fixed n, $n \geq 1$, the hazard rate h_W of $W = X_{L(n+1)}/X_{L(n)}$ is the same as the hazard rate h of X_n or the hazard rate h_V of $V = X_{L(n+2)}/X_{L(n+1)}$.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with cumulative density function(cdf) $F(x)$ and probability density function(pdf) $f(x)$. Let $Y_n = \max(\min)\{X_1, X_2, \ldots, X_n\}$ for $n \geq 1$. We say X_j is an upper(lower) record value of this sequence, if $Y_j > (<) Y_{j-1}$ for $j > 1$. By definition, X_1 is an upper as well as a lower record value. The indices at which the upper record values occur are given by the record times $\{U(n), n \geq 1\}$, where $U(n) = \min\{j \mid j > U(n-1), X_j > X_{U(n-1)}, n \geq 2\}$ with $U(1) = 1$. We assume that all upper record values $X_{U(i)}$ for $i \geq 1$ occur at a sequence $\{X_n, n \geq 1\}$ of i.i.d. random variables.

Let F be the distribution function of a nonnegative random variable, then we will say F is "new better than used"(NBU), if $\bar{F}(xy) \leq \bar{F}(x)\bar{F}(y)$, x, $y > 1$, and F is "new worse than used"(NWU), if $\bar{F}(xy) \geq \bar{F}(x)\bar{F}(y)$, x, $y > 1$. We say the F belongs to the class C_1 if F is either NBU or NWU.

If $F(x)$ has density $f(x)$, the ratio $h(x) = f(x)/F(x)$, for $F(x) > 0$, is called the hazard rate in the lower record values. We say F belongs to the class C_2 if $h(uw)u \leq h(w)$ or $h(uw)u \geq h(w)$. Also, we say F belongs to the class C_3 if $h(uw)u \leq \alpha$ or $h(uw)u \geq \alpha$.

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We say the random variable $X \in POW(a, \alpha)$ if the corresponding probability cdf $F(x)$ of X is of the form

$$F(x) = \begin{cases} \left(\frac{x}{a}\right)^{\alpha}, & 0 \le x \le a, \ \alpha > 0 \\ 0, & otherwise. \end{cases}$$

In [2], Ahsanullah characterized that X_k belongs to the class C_1 and $X_{U(n)} - X_{U(m)}$ and $X_{U(m)}$, $0 < m < n$ are identically distributed, then $X_k \in E(x, \sigma)$. In [3], Ahsanullah et al. obtained the characterization of Pareto distribution by the hazard rate of upper record values. Also, we can find more details on characterizations under assumption of identical distribution in [4].

In this paper, we obtain characterizations of the power distribution by the identical harzard rate of lower record values.

2. Results

THEOREM 2.1. *Let $\{X_n, \ n \ge 1\}$ be a sequence of independent and identically distributed non-negative random variables with absolutely continuous cdf $F(x)$ and pdf $f(x)$ with $F(1) = 1$. Let F belongs to class C_2. Then X_n has the power distribution if and only if for some fixed n, $n \ge 1$, the hazard rate h_W of $W = X_{L(n+1)}/X_{L(n)}$ is the same as the hazard rate h of X_n.*

THEOREM 2.2. *Let $\{X_n, \ n \ge 1\}$ be a sequence of independent and identically distributed non-negative random variables with absolutely continuous cdf $F(x)$ and pdf $f(x)$ with $F(1) = 1$. Let F belongs to class C_3. Then X_n has the power distribution if and only if for some fixed n, $n \ge 1$, the hazard rate h_W of $W = X_{L(n+1)}/X_{L(n)}$ is the same as the hazard rate h_V of $V = X_{L(n+2)}/X_{L(n+1)}$.*

3. Proofs

Proof of Theorem 2.1. If $X_n \in POW(1, \alpha)$ for $\alpha > 0$, then it can be easily shown that $h_W = h$. We have to prove the converse.

We can write the joint pdf of $X_{L(n)}$ and $X_{L(n+1)}$ as

$$f_{n,n+1}(x, y) = \frac{\{H(x)\}^{n-1}}{\Gamma(n)} h(x) f(y), \ 0 \le y < x \le 1$$

where $H(x) = -lnF(x)$ and $h(x) = -\frac{d}{dx}H(x)$.

Let us use the transformation $U = X_{L(n)}$ and $W = X_{L(n+1)}/X_{L(n)}$. Jacobian of the transformation is u. Thus we can write the joint pdf $f_{U,W}(u,w)$ of U and W as

$$f_{U,W}(u,w) = \frac{\{H(u)\}^{n-1}}{\Gamma(n)} h(u) f(uw) u \tag{3.1}$$

for $0 \le u \le 1$, $0 \le w < 1$. Thus by (3.1), we can write

$$h_W(w) = \frac{\int_0^1 \{H(u)\}^{n-1} h(u) f(uw) u du}{\int_0^1 \{H(u)\}^{n-1} h(u) F(uw) du}$$

for $0 \le u \le 1$, $0 \le w < 1$. Since $h_W(w) = h(w)$ for all w, we have

$$\frac{\int_0^1 \{H(u)\}^{n-1} h(u) f(uw) u du}{\int_0^1 \{H(u)\}^{n-1} h(u) F(uw) du} = \frac{f(w)}{F(w)} \tag{3.2}$$

for all $0 \le u \le 1$, $0 \le w < 1$. Now simplifying (3.2), we have

$$\int_0^1 H(u)^{n-1} h(u) F(w) F(uw) \{h(uw)u - h(w)\} du = 0$$

for $0 \le u \le 1$. Thus if $F \in C_2$, then we get the following equation

$$h(uw)u = h(w) \tag{3.3}$$

for almost all u, $0 \le u \le 1$ and any fixed w, $0 \le w < 1$. Integrating (3.3) with respect to w from w_1 to 1 and simplifying, we get

$$F(uw_1) = F(u)F(w_1) \tag{3.4}$$

for all $0 \le u \le 1$, $0 \le w_1 \le 1$. By the theory of functional equations (see [1]), the only continuous solution of (3.4) with the boundary conditions $F(0) = 0$ and $F(1) = 1$ is $F(x) = x^\alpha$ for all $0 \le x \le 1$ and $\alpha > 0$. This completes the proof. □

Proof of Theorem 2.2. If $X_n \in POW(1, \alpha)$ for $\alpha > 0$, then it can be easily shown that $h_W = h_V$. We have to prove the converse.

We can write the joint pdf of $X_{L(n)}$ and $X_{L(n+1)}$ as

$$f_{n,n+1}(x,y) = \frac{\{H(x)\}^{n-1}}{\Gamma(n)} h(x) f(y),\ 0 \le y < x \le 1.$$

Let us use the transformation $U = X_{L(n)}$ and $W = X_{L(n+1)}/X_{L(n)}$. Jacobian of the transformation is u. Thus we can write the joint pdf $f_{U,W}(u,w)$ of U and W as

$$f_{U,W}(u,w) = \frac{\{H(u)\}^{n-1}}{\Gamma(n)} h(u) f(uw) u \tag{3.5}$$

for $0 \le u \le 1$, $0 \le w < 1$. Thus by (3.5), we can write

$$h_W(w) = \frac{\int_0^1 \{H(u)\}^{n-1} h(u) f(uw) u du}{\int_0^1 \{H(u)\}^{n-1} h(u) F(uw) du}$$

for $0 \le u \le 1$, $0 \le w < 1$. Since $h_W = h_V$ for all w, we have

$$\text{(3.6)} \qquad \frac{\int_0^1 \{H(u)\}^{n-1} h(u) f(uw) u du}{\int_0^1 \{H(u)\}^{n-1} h(u) F(uw) du} = \frac{\int_0^1 \{H(u)\}^{n} h(u) f(uw) u du}{\int_0^1 \{H(u)\}^{n} h(u) F(uw) du} = \alpha$$

for all $0 \le u \le 1$, $0 \le w < 1$, $\alpha > 0$.

From (3.6), we have

$$\int_0^1 H(u)^k f(uw) u du = \alpha \int_0^1 H(u)^k F(uw) du$$

for $k = n-1, n$. Now simplifying (3.6), we have

$$\int_0^1 H(u)^k h(u) F(uw) \{h(uw)u - \alpha\} du = 0$$

for all $0 \le u \le 1$. Thus if $F \in C_3$, then we get the following equation

$$\text{(3.7)} \qquad h(uw)u = \alpha$$

for almost all u, $0 \le u \le 1$ and any fixed w, $0 \le w < 1$. Now letting $w \to 1$, we obtain from (3.7)

$$\text{(3.8)} \qquad h(u)u = \alpha \ \text{ i.e. } \ f(u)/F(u) = \alpha/u$$

for almost all u, $0 \le u \le 1$ and $\alpha = f(1) > 0$. Integrating (3.8) and using the boundary conditions $F(0) = 0$ and $F(1) = 1$, we obtain $F(x) = x^\alpha$ for all $0 \le x \le 1$ and $\alpha > 0$. This completes the proof. □

References

[1] J. Aczel, *Lectures on Functional Equation and Their Applications*, Academic Press, Newyork, 1966.

[2] M. Ahsanuallah, *Record values and the exponential distribution*, Ann. Inst. Statist. Math. **30** (1978), Part A, 429-433.

[3] M. Ahsanuallah, *A note the characterization of Pareto distribution by the hazard rate of upper record values*, Pak. J. Statist. **29** (2013), 447-452.

[4] M. Ahsanuallah and Z. Raqab, *Bounds and Characterizations of Record Statistics*, Nova science Publishers, Inc. (2004).

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